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THE ZERO–TWO LAW FOR POSITIVE CONTRACTIONS IS VALID IN ALL BANACH LATTICES

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ABSTRACT

It is shown that the zero–two law for positive contractions, as established recently by Y. Katznelson and L. Tzafriri (1986), holds in all Banach lattices.

1. In a recent paper [2], Y. Katznelson and L. Tzafriri have shown that for an arbitrary power-bounded operator T on a complex B-space, one has $\lim_n \| T^n - T^{n+1} \| = 0$ iff the spectrum $\sigma(T)$ intersects the unit circle $\Gamma = \{z \in \mathbf{C} : |z| = 1\}$ at most in $z = 1$. As an application of their analysis, they have extended the so-called zero–two law established earlier in [1], [3], and [5] to a large class of complex Banach lattices (essentially Köthe function spaces satisfying a so-called δ-condition, see [2]). The result [2, Thm. 6] asserts that if T is a positive, linear contraction on such a space X then either $\lim_n \| T^n - T^{n+1} \| = 0$ or else the modulus $| T^n - T^{n+1} |$ has norm 2 for all $n = 0, 1, 2, \ldots$.

The purpose of this note is to show that this latter alternative actually holds for positive (linear) contractions on an arbitrary Banach lattice X, with one necessary modification: Since on an arbitrary Banach lattice, the modulus of $T^n - T^{n+1}$ does not necessarily exist (see [4, IV.1]), the quantity to be substituted for its norm is the so-called regular norm of $T^n - T^{n+1}$, denoted by $\| T^n - T^{n+1} \|_r$. (We recall that for any operator R on X which is the difference of two positive operators, the regular norm is defined to be

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$$\| R \|_r = \inf \| S_1 + S_2 \| ,$$

the infimum being taken over all representations $R = S_1 - S_2$ with $0 \leqq S_i \in \mathscr{L}(X)$ $(i = 1, 2)$. In case $|R|$ exists then so do $R^+ = R \vee 0$ and $R^- = (-R) \vee 0$, and $\| R \|_r = \| R^+ + R^- \|$. For further details, see [4, IV.1].) Thus we will prove the following extension of [2, Thm. 6].

THEOREM. *Let T be a positive contraction on an arbitrary Banach lattice. Then either* $\lim_n \| T^n - T^{n+1} \| = 0$, *or else* $\| T^n - T^{n+1} \|_r = 2$ *for all integers* $n \geqq 0$.

2. We will generally follow the terminology and notation of [4]; in particular, see [4, II.11] for the relation between real and complex Banach lattices. The essence of our assertion is contained in this lemma.

LEMMA. *Let X be any complex Banach lattice. Suppose T is a positive (linear) contraction on X with an eigenvalue $\alpha \neq 1$, $|\alpha| = 1$, and an eigenvector f satisfying $\alpha f = Tf$, $|f| = T|f|$. Then for all integers $n \geqq 0$ we have* $\| T^n - T^{n+1} \|_r = 2$.

PROOF. Consider the principal (complex) lattice ideal $X_{|f|}$ of X; by Kakutani's theorem [4, II.7.4] we can and will identify $X_{|f|}$ with the space $C(K)$ of continuous complex functions on some compact space K, such that $|f|$ becomes the unit e of $C(K)$. Obviously, T leaves $C(K)$ invariant and each power T^n $(n \geqq 0)$ defines a Markov operator on $C(K)$ with unimodular eigenfunction f. Throughout the proof, the integer $n \geqq 0$ will be fixed.

Now let $s \in K$ be fixed also. Evaluation of $T^n f$ and $T^{n+1} f$ at s is given by $T^n f(s) = \int f(t) d\mu_s(t)$ and $T^{n+1} f(s) = \int f(t) d\nu_s(t)$, where $\mu_s = (T')^n \delta_s$, $\nu_s = (T')^{n+1} \delta_s$ (δ_s Dirac measure at $s \in K$) are Radon probability measures on K. Assume now that $f(s) = 1$. From $\alpha^n f = T^n f$, $\alpha^{n+1} f = T^{n+1} f$ and a standard argument (cf. [4, V.4.2]) it now follows that $f(t) \equiv \alpha^n$ on the (closed) support A_s of μ_s, and that $f(t) \equiv \alpha^{n+1}$ on the support B_s of ν_s. In particular, A_s and B_s are disjoint, and by Urysohn's theorem there exists a function $k_s \in C(K)$, $|k_s| \leqq e$, which takes the values α^{-n} and $-\alpha^{-n-1}$ throughout the sets A_s and B_s, respectively. This implies

$$(*) \qquad (T^n - T^{n+1})(k_s)(s) = 2.$$

Now let $T^n - T^{n+1} = S_1 - S_2$ be any decomposition where $0 \leqq S_i \in \mathscr{L}(X)$ $(i = 1, 2)$. Then we have

$$|(T^n - T^{n+1})k_s| = |S_1 k_s - S_2 k_s| \leqq (S_1 + S_2)e.$$

Now select, for each $s \in K$, a function k_s as above. Then for all $s \in K$, $|k_s| \leqq e$ in X. On the other hand, because of $|(T^n - T^{n+1})k_s| \leqq 2e$ and since $C(K)$ is an ideal of X, it follows from (*) that $\sup_s |(T^n - T^{n+1})k_s|$ exists in X and is $\leqq (S_1 + S_2)e$. Thus from (*) we obtain

$$2e = \sup_{s \in K} |(T^n - T^{n+1})k_s| \leqq (S_1 + S_2)e.$$

This implies $\| S_1 + S_2 \| \geqq 2$ and hence $\| T^n - T^{n+1} \|_r = 2$, since T^n and T^{n+1} are contractions.

3. The proof of the Theorem will now consist of several technical steps showing that we can place ourselves in the situation of the preceding Lemma. Let us note first that by [2, Thm. 1] we only have to show that the presence of a unimodular number $\alpha \neq 1$ in $\sigma(T)$ implies $\| T^n - T^{n+1} \|_r = 2$ for all n.

(i) As in [2, p. 322] we first replace X by an ultraproduct (or simply an F-product) Y which again is a Banach lattice (for details, see [4, V.1]). For the positive contraction $S \in \mathscr{L}(Y)$ corresponding to $T \in \mathscr{L}(X)$, α is an eigenvalue and we have $\alpha f = Sf$, $|f| \leqq S|f|$. It is easy to see but important to note that $\| S^n - S^{n+1} \|_r \leqq \| T^n - T^{n+1} \|_r$.

(ii) We next consider the second dual Y'' and the (unique weak* continuous) extension $\bar{S}$ of S to Y''. Again we note that $\| \bar{S}^n - \bar{S}^{n+1} \|_r \leqq \| S^n - S^{n+1} \|_r$.

(iii) Now clearly $|f| \leqq \bar{S}|f|$ and, since $\alpha \in \sigma(\bar{S})$ and $\bar{S}$ is a contraction, we have $r(\bar{S}) = 1$ for the spectral radius of $\bar{S}$. Thus we can apply Lemma [4, V.4.8] asserting the existence of a positive fixed vector ϕ of $\bar{S}'$ (indeed, of S') such that $\phi(|f|) > 0$. Considering the quotient Z of Y'' over the $\bar{S}$-invariant ideal $J = \{g \in Y'' : \phi(|g|) = 0\}$, we obtain an operator (positive contraction) $\tilde{S}$ on Z satisfying $\alpha \hat{f} = \tilde{S}\hat{f}$, $|\hat{f}| = \tilde{S}|\hat{f}|$ in obvious notation. Thus by the Lemma proved above, we obtain $\| \tilde{S}^n - \tilde{S}^{n+1} \|_r = 2$ for all n.

The proof will now be complete if we can show that, again,

$$\| \tilde{S}^n - \tilde{S}^{n+1} \|_r \leqq \| \bar{S}^n - \bar{S}^{n+1} \|_r.$$

But since Y'' is Dedekind complete, the modulus $V_n = |\bar{S}^n - \bar{S}^{n+1}|$ exists in

$\mathscr{L}(Y'')$; because of $V_n \leqq \tilde{S}^n + \tilde{S}^{n+1}$, V_n also leaves the ideal J invariant. Hence V_n induces a positive operator $\tilde{V}_n$ on Z and we finally have

$$\| \tilde{S}^n - \tilde{S}^{n+1} \|_r \leqq \| \tilde{V}_n \| \leqq \| V_n \| = \| \tilde{S}^n - \tilde{S}^{n+1} \|_r.$$

4. We remark in conclusion that by using the preceding and similar techniques, it can be shown that [2, Thm. 10] is equally valid in arbitrary Banach lattices. Precisely:

If S, T are commuting positive (linear) contractions on an arbitrary Banach lattice X then either $\lim_n \| (S - T)S^nT^n \| = 0$, *or else* $\| (S - T)S^nT^n \|_r = 2$ *for all integers* $n \geqq 0$.

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